Weighted shifts induced by positive definite sequences (Research on structure of operators using operator means and related topics)

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1 Introduction and preliminaries

This article is based on papers [7], [8] and [9]. The reader can find the details in them. Some results in this article which come from [8] will be appeared in some other journal.

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. We denote by $[A, B] := AB-BA$ the *commutator* of A and B in $\mathcal{L}(\mathcal{H})$. Let $\mathbb{N}$ [resp., $\mathbb{Z}_+$] be the set of positive integers [resp., nonnegative integers]. We write $\mathbb{R}$ [resp., $\mathbb{R}_+, \mathbb{C}$] for the set of real [resp. nonnegative real, complex] numbers and let $\mathbb{R}_+^0 := \mathbb{R}_+ \backslash \{0\}$.

An operator T in $\mathcal{L}(\mathcal{H})$ is *subnormal* if it is (unitarily equivalent to) the restriction of a normal operator to an invariant subspace, and *hyponormal* if $[T^*, T] \geq 0$. It is well-known that an operator T in $\mathcal{L}(\mathcal{H})$ is subnormal if and only if $\sum_{0\leq i,j\leq n} \langle T^{*i}T^j h_i, h_j\rangle \geq 0$ for all $h_i, h_j \in \mathcal{H}$ and $n \in \mathbb{N}$ ([1],[10]). For a fixed $n \in \mathbb{N}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is *n-hyponormal* if $\sum_{0\leq i,j\leq n} \langle T^{*i}T^j h_i, h_j\rangle \geq 0$ for all $h_i, h_j \in \mathcal{H}$. Thus $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if T is n-hyponormal for all $n \in \mathbb{N}$. Obviously, the implications "subnormal $\Rightarrow \cdots \Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal" hold, and it is well-known that each converse is not always true ([2],[11]).

Given a sequence $\{\gamma_n\}_{n=0}^\infty \subset \mathbb{R}_+^0$, *the Stieltjes moment problem* entails determining whether there exists, and finding when it does, a positive Borel measure μ on $\mathbb{R}$ supported on $\mathbb{R}_+$ such that

$$\gamma_n = \int_{\mathbb{R}_+} t^n \, d\mu(t), \qquad n \in \mathbb{Z}_+.$$

Such a sequence $\{\gamma_n\}_{n=0}^\infty$ [resp., measure μ] is called a *Stieltjes moment sequence* [resp., *Stieltjes moment measure*]. Furthermore, it is well-known that $\{\gamma_n\}_{n=0}^\infty$ is a Stieltjes moment sequence if and only if the two infinite matrices $(\gamma_{i+j})_{0\leq i,j<\infty}$ and $(\gamma_{i+j+1})_{0\leq i,j<\infty}$ are positive (cf. [13]).

Given a sequence $\{\gamma_n\}_{n=0}^{\infty} \subset \mathbb{R}$, the analogous *Hamburger moment problem* relaxes the requirement to a positive Borel measure μ supported merely on $\mathbb{R}$ such that

$$\gamma_n = \int_{\mathbb{R}} t^n \, d\mu(t), \qquad n \in \mathbb{Z}_+.$$

If this is possible the sequence $\{\gamma_n\}_{n=0}^{\infty}$ and measure μ are called a *Hamburger moment sequence* and a *Hamburger moment measure*, respectively. It follows from [13] that $\{\gamma_n\}_{n=0}^{\infty}$ is a Hamburger moment sequence if and only if $(\gamma_{i+j})_{0\leq i,j<\infty}$ is positive.

We set some notation for the standard testing ground of weighted shift operators. Let $\{e_i\}_{i\in\mathbb{Z}_+}$ be the canonical orthonormal basis for $\ell^2(\mathbb{Z}_+)$. Given a weight sequence $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ of positive real numbers, we define the weighted shift W_α by $W_\alpha e_k = \alpha_k e_{k+1}$ and extend by linearity. We define the moment sequence $\{\gamma_i\}_{i=0}^{\infty}$ by

$$\gamma_0 = 1; \quad \gamma_i := \alpha_0^2 \cdots \alpha_{i-1}^2, \qquad i \in \mathbb{N}.$$

The organization of this paper is as follows. In Section 2, we give basic definitions, constructions, and examples. In Section 3, we discuss relationships among subnormality, Hamburger-type property, properties $H(n)$ and $\widetilde{H}(n)$, and obtain some results distinguishing the various classes under study. In Section 4, we consider flatness (the propagation of equal adjacent weights to some or all other weights) and in In Section 5 we consider completion problems (indicating, for example how to complete three initial weights and when the resulting completion results in a shift with positive weights) and finally give a remark. Section 6 we consider matters of backward n-step extensions and perturbations.

Some of the calculations in this paper were aided by use of the software tool Mathematica (see [12]).

2 Basic constructions

Let $\alpha = \{\alpha_k\}_{k=0}^{\infty}$ be a sequence of positive real numbers and let W_α be the associated weighted shift with weight sequence α. For $k, n \in \mathbb{Z}_+$, we set

$$M_n(k) = \begin{pmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+n} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+n} & \gamma_{k+n+1} & \cdots & \gamma_{k+2n} \end{pmatrix}.$$

Note that the matrix is of size $(n+1)$ by $(n+1)$, and is, in fact, the standard matrix considered for n-hyponormality of weighted shifts.

Definition 2.1 A weighted shift W_α has *property* $H(n)$ [resp., *property* $\widetilde{H}(n)$] if $M_n(k) \geq 0$ for all $k = 0, 2, 4, ...$ [resp., $M_n(k) \geq 0$ for all $k = 1, 3, 5, ...$]. And W_α

has *property* $H(\infty)$ [resp., property $\widetilde{H}(\infty)$] if it has property $H(n)$ [resp., property $\widetilde{H}(n)$] for all $n \in \mathbb{N}$. In particular, we say that W_α is a *Hamburger-type weighted shift* if W_α has property $H(\infty)$.

Note that, for some $n \in \mathbb{N}$, W_α is n-hyponormal if and only if W_α has both properties $H(n)$ and $\widetilde{H}(n)$. Therefore W_α is subnormal if and only if it has properties $H(n)$ and $\widetilde{H}(n)$ for all $n \in \mathbb{N}$. Moreover, elementary computations show that W_α has property $H(1)$ [resp., property $\widetilde{H}(1)$] if and only if $\alpha_{2n+1} \geq \alpha_{2n}$ [resp., $\alpha_{2n+2} \geq \alpha_{2n+1}$] for all $n \in \mathbb{Z}_+$. Obviously, then, the properties $H(n)$ and $\widetilde{H}(n)$ are distinct for each n and distinct from n-hyponormality, but note that the well-known fact that W_α is hyponormal (which is 1-hyponormal) if and only if its weights are weakly increasing splits neatly into two requirements related to the properties $H(1)$ and $\widetilde{H}(1)$. It turns out that, unsurprisingly, even property $H(\infty)$ does not imply either $\widetilde{H}(n)$ or n-hyponormality for any n (see Example 2.2).

We emphasize the fact that if W_α is Hamburger-type then the sequence $\{\gamma_n\}_{n=0}^{\infty} \geq 0$ is a Hamburger moment sequence, but under our convention of positive weights it carries the additional information that each γ_n is positive. If W_α is Hamburger-type we will sometimes call the measure associated to W_α the *Hamburger measure* μ.

We turn to some examples showing certain classes are distinct.

Example 2.2 Consider $\alpha : \alpha_n = \sqrt{\frac{2^{n+1}+(-1)^{n+1}}{2^n+(-1)^n}}$ $(n \geq 0)$. Observe that the measure $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_2$ satisfies

$$\gamma_n = \frac{1}{2}(2^n + (-1)^n) = \int_{\mathbb{R}} t^n d\mu(t), \qquad n \in \mathbb{Z}_+.$$

Hence W_α has property $H(\infty)$. But since $\det(\gamma_{i+j+1})_{i,j=0}^{1} = -\frac{9}{2} < 0$, W_α does not have property $\widetilde{H}(n)$ for any $n \in \mathbb{N}$. So W_α is not n-hyponormal for any $n \in \mathbb{N}$. This example shows that in general the properties $H(\infty)$ and $\widetilde{H}(n)$ (and thus certainly $\widetilde{H}(\infty)$ and subnormality) are different.

In general, property $H(n)$ does not imply property $H(n+1)$ for any $n \in \mathbb{N}$.

Example 2.3 Let $\alpha : \sqrt{x}, \sqrt{\frac{k+1}{k+2}}$ $(k \geq 1)$ and let W_α be the associated weighted shift. By the techniques in the proof of [4, Th. 4] (and see originally [2, Prop.7]), we obtain that W_α has property $H(n)$ if and only if $0 \leq x \leq \frac{(n+1)^2}{2n(n+2)}$ for $n \in \mathbb{N}$. (In fact, in this case, property $H(n)$ for W_α is equivalent to n-hyponormality and the sole new thing to check is that what is in play is the property $H(n)$ portion of n-hyponormality.)

Some improved examples related to properties $H(n)$, $\widetilde{H}(n)$ and n-hyponormality will be discussed in the next section.

We pause to record an easy fact motivated by the example below of a "backward 1-step extension."

Example 2.4 (Continued from Example 2.2) Let us consider a weight sequence

$$\alpha(x) : \alpha_0 = \sqrt{x},\ \alpha_n = \sqrt{\frac{2^{n+1} + (-1)^{n+1}}{2^n + (-1)^n}}, \quad n \geq 1,$$

where x is a positive real variable. Then a direct computation shows that

(i) $W_{\alpha(x)}$ has property $H(1)$ if and only if $0 < x \leq 5$,

(ii) $W_{\alpha(x)}$ has property $H(n)$ for some $n \geq 2$ [or, for all $n \geq 1$] if and only if $0 < x \leq \frac{1}{2}$, which is equivalent to $W_{\alpha(x)}$ has property $H(\infty)$,

3 Distinctions

Before considering distinctions of properties $H(n)$ and $\widetilde{H}(n)$, we characterize those properties for our purpose.

Theorem 3.1 *Let* $W_\alpha := W_{\alpha(x,y,z)}$ *be a weighed shift with weight sequence* $\alpha(x,y,z)$: $\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{4/5}, \sqrt{\frac{k+1}{k+2}}$ $(k \geq 4)$. *Then the following assertions hold:*

(i) $W_{\alpha(x,y,z)}$ *has property* $H(1)$ *if and only if* $x \leq y$ *and* $z \leq \frac{4}{5}$,

(ii) $W_{\alpha(x,y,z)}$ *has property* $H(2)$ *if and only if* $-5xy^2 - 4xz + 4yz + 10xyz - 5yz^2 \geq 0$ *and* $z \leq \frac{25}{33}$,

(iii) *For* $n \geq 3$, $W_{\alpha(x,y,z)}$ *has property* $H(n)$ *if and only if the following two conditions hold:*

(a) $\Theta_1(z;n) \geq 0$, *where* $\Theta_1(z;n) := \frac{3(n+1)^2(n+2)^2(n+3)^2}{4((n+1)^2(n+2)^2(n+3)^2-36)} - z$,

(b) $\Theta_2(x,y,z;n) := \lambda_n + \mu_n z + 4xyz^2\psi_n(y,z) \geq 0$, *where*

$$\psi_n(y,z) = \Delta_n^{(1)} + 2\delta\Delta_n^{(2)} + \delta\Delta_n^{(3)} - 2\delta^2\Delta_n^{(4)} - 2\epsilon\Delta_n^{(5)} - \delta^2\Delta_n^{(6)} + 2\epsilon\delta\Delta_n^{(7)} - \delta^3 d_{n-3}^{(7)} - \epsilon^2 d_{n-2}^{(5)}$$

with $\varepsilon := \epsilon(y,z) = \frac{1}{4yz} - \frac{1}{2}$, $\delta := \delta(z) = \frac{1}{4z} - \frac{1}{3}$, $\mu_n = \mu_n^{(1)}(6 - n^3 - 3n^2 - 2n)$, *and*

$$\lambda_n = \frac{(G(n))^2\,(G(n+4))^2}{48G(2n+3)},$$

$$\mu_n^{(1)} = \frac{G(n)^2 G(n+3) G(n+4)}{G(2n+3)} \cdot \frac{(n-1)!\,(6 + n(n+1)(n+2))}{36n(n+1)(n+2)}.$$

Theorem 3.2 *Let* $\alpha(x,y,z)$: $\sqrt{x}, \sqrt{y}, \sqrt{z}, \sqrt{4/5}, \alpha_k = \sqrt{\frac{k+1}{k+2}}$ $(k \geq 4)$ *with positive variables. Then the following assertions hold:*

(i) $W_{\alpha(x,y,z)}$ *has property* $\widetilde{H}(1)$ *if and only if* $0 < y \le z$,

(ii) $W_{\alpha(x,y,z)}$ *has property* $\widetilde{H}(2)$ *if and only if* $2z - 5y(10 - 24z + 15z^2) \ge 0$,

(iii) *For* $n \ge 3$, $W_{\alpha(x,y,z)}$ *has property* $\widetilde{H}(n)$ *if and only if* $y < z$ *and*

$$\Omega(y, z; n) := \frac{5184(n+1)^2(n+2)^2 z}{n(n+3)(A_n z^2 + B_n z + C_n)} - y \ge 0, \tag{3.2}$$

where

$$A_n = 16(n-1)(n+4)(n^8 + 12n^7 + 66n^6 + 216n^5 + 477n^4 + 756n^3 + 680n^2 + 96n - 360),$$
$$B_n = -24(n-1)(n+1)^2(n+2)^2(n+4)(n^4 + 6n^3 + 17n^2 + 24n + 36),$$
$$C_n = 9n(n+1)^4(n+2)^4(n+3).$$

With helping Theorems 3.1 and 3.2, We give a family of examples showing that the properties $H(n)$, $\widetilde{H}(n)$ and n-hyponormality are distinct.

Examples for distinctions. We now give a family of examples showing that the properties $H(n)$, $\widetilde{H}(n)$ and n-hyponormality are distinct. Instead of the sequence $\alpha(x, y, z)$ in Theorems 3.1 and 3.2 we consider the sequence $\alpha(x, y, \frac{3}{4})$ with two variables x and y, which allows for relatively easy visualization of relevant regions in $\mathbb{R}^2$. Let

$$\alpha(x, y) : \sqrt{x}, \sqrt{y}, \left\{ \sqrt{\frac{k+1}{k+2}} \right\}_{k=2}^{\infty}$$

be a weight sequence with positive real variables x, y and let $W_{\alpha(x,y)}$ be the associated weighted shift with weight sequence $\alpha(x, y)$. We may assume $n \ge 3$ because the cases for $n = 1$ and $n = 2$ are simple: $W_{\alpha(x,y,\frac{3}{4})}$ has property $H(1)$ if and only if $x \le y$, property $H(2)$ if and only if $3y - 48x + 120xy - 80xy^2 \ge 0$, property $\widetilde{H}(1)$ if and only if $y \le \frac{3}{4}$, and has property $\widetilde{H}(2)$ if and only if $y \le \frac{24}{35}$.

First, we consider the properties $H(n)$ for $W_{\alpha(x,y)}$ for $n \ge 3$. According to Theorem 3.1 with $z = \frac{3}{4}$, we have that $W_{\alpha(x,y)}$ has property $H(n)$ if and only if it satisfies the following two conditions:

$$\Theta_1\left(\frac{3}{4}; n\right) = \frac{3}{4}\left[\frac{(n+1)^2(n+2)^2(n+3)^2}{(n+1)^2(n+2)^2(n+3)^2 - 36} - 1\right] \ge 0,$$

$$\Theta_2\left(x, y, \frac{3}{4}; n\right) = \lambda_n + \frac{3}{4}\mu_n + \frac{9}{4}xy\psi_n(y, \frac{3}{4}) \ge 0,$$

where $\psi_n(y, \frac{3}{4}) = \Delta_n^{(1)} - 2\epsilon\Delta_n^{(5)} - \epsilon^2 d_{n-2}^{(5)}$ with $\varepsilon := \epsilon(y, \frac{3}{4}) = \frac{1}{3y} - \frac{1}{2}$ and $\delta = 0$. By some computations, we can see that

$$\Theta_2\left(x, y, \frac{3}{4}; n\right) = \frac{(G(n+1))^2 (G(n+3))^2}{G(2n+3)} \frac{n(n+2)}{192(n+1)^2} \frac{x}{y} \theta(x, y; n),$$

where

$$\theta(x, y; n) = -A_n y^2 - B_n y - C_n + \frac{144(n+1)^2}{n(n+2)} \frac{y}{x}$$

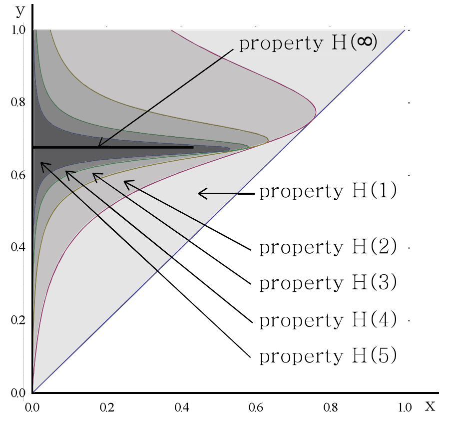
with

$$\begin{aligned} A_n &= 9(n-1)(n+3)(n^4 + 4n^3 + 9n^2 + 10n - 8), \\ B_n &= -12(n-1)(n+1)^2(n+3)(n^2 + 2n + 4), \\ C_n &= 4n(n+1)^4(n+2). \end{aligned}$$

Since $\Theta_1\left(\frac{3}{4}; n\right) \geq 0$ is clear, we obtain that $W_{\alpha(x,y)}$ has property $H(n)$ if and only if $\theta(x, y; n) \geq 0$, i.e.,

$$0 < x \leq \frac{144(1+n)^2 y}{n(n+2)(A_n y^2 + B_n y + C_n)}. \tag{3.3}$$

According to (3.3), we can see that $W_{\alpha(x,y)}$ has property $H(\infty)$ if and only if $y = \frac{2}{3}$ and $0 < x \leq \frac{1}{2}$.



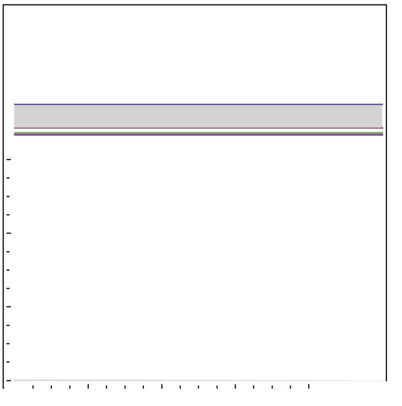


Figure 3.1. Illustration of properties $H(n)$

in (3.4) is strictly negative, the ranges for the properties $\widetilde{H}(n)$ are strictly decreasing in n.)

And also, combining conditions (3.3) and (3.4), we have that $W_{\alpha(x,y)}$ is n-hyponormal if and only if the following two conditions hold:

$$0 < y \leq \frac{2(n+1)^2(n+2)^2}{3(n^2+3n)(n^2+3n+4)} \quad \text{and} \quad 0 < x \leq \frac{144(1+n)^2 y}{n(n+2)(A_n y^2 + B_n y + C_n)}.$$

Hence $W_{\alpha(x,y)}$ is subnormal if and only if $y = \frac{2}{3}$ and $0 < x \leq \frac{1}{2}$. Figures 3.1, 3.2, and 3.3 illustrate distinctions among the properties $H(n)$, $\widetilde{H}(n)$ and n-hyponormality of $W_{\alpha(x,y)}$ by regions in $\mathbb{R}^2$. Note that the region of n-hyponormality is the common part of the regions corresponding to properties $H(n)$ and $\widetilde{H}(n)$. Hence Figure 3.1 and Figure 3.2 provide Figure 3.3.

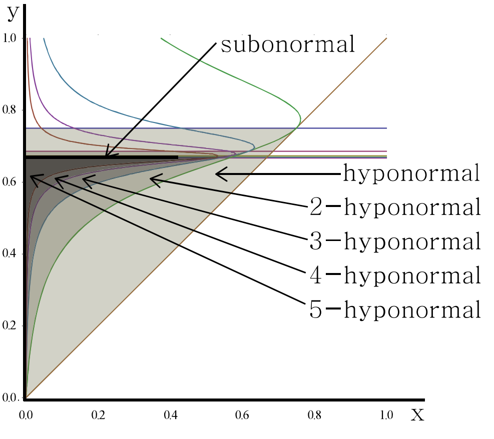


Figure 3.3. Illustration of n-hyponormality

4 Flatness

We now consider the flatness of weighted shifts with property $H(n)$. As we discussed in the introduction, if W_α is subnormal (even 2-hyponormal) with $\alpha_n = \alpha_{n+1}$ ($n \in \mathbb{Z}_+$), then $\alpha_1 = \alpha_2 = \cdots$. But this flatness property need not hold in weighted shifts with property $H(2)$ as we show next.

Example 4.1 Let $\alpha(x)$ be given by

$$\sqrt{x}, \sqrt{31/17}, \sqrt{31/17}, \sqrt{31/17}, \sqrt{31/17}, \sqrt{65/31}, \alpha_n = \sqrt{\frac{2^{n+1}+(-1)^{n+1}}{2^n+(-1)^n}} \ \ (n \geq 6).$$

(Recall that the tail of this sequence arises from $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_2$ as in Example2.2 and therefore $W_\alpha|_{\vee\{e_i\}_{i=4}^\infty}$ is of Hamburger-type.) One computes that $\det M_1(0) = \gamma_0\gamma_1(31/17 - x)$, $\det M_2(0) = 0$, $\det M_1(2) = 0$, and $\det M_2(2) = 0$. Positivity of other $M_1(2k)$ and $M_2(2k)$ is ensured because we have a Hamburger-type tail. Observe that $W_{\alpha(x)}$ has property $H(2)$ if and only if $0 < x \leq 31/17$. Thus, a weighted shift with property $H(2)$ may have five equal (successive) weights without being flat.

It turns out that property $H(n)$ for any $n \geq 3$ is sufficient to guarantee flatness if the two successive equal weights begin at an even index.

Theorem 4.2 *Let W_α be a weighted shift with property $H(3)$. If $\alpha_{2n} = \alpha_{2n+1}$ for some $n \in \mathbb{Z}_+$, then $\alpha_1 = \alpha_2 = \cdots$.*

Observe that in the work above we have actually proved along the way the following limited "propagation" result (and compare Example 4.1).

Corollary 4.3 *Let W_α be a weighted shift with property $H(2)$. If $\alpha_{2n} = \alpha_{2n+1}$ for some $n \in \mathbb{Z}_+$, then $\alpha_{2n-1} = \alpha_{2n} = \alpha_{2n+1} = \alpha_{2n+2}$.*

We now consider the jumping flatness of W_α whose definition appears in [8].

Theorem 4.4 *Let W_α be a weighted shift with property $H(4)$. If $\alpha_{2n} = \alpha_{2n+2}$ and $\alpha_{2n+1} = \alpha_{2n+3}$ for some $n \in \mathbb{Z}_+$, then $\alpha_1 = \alpha_3 = \alpha_5 = \cdots$ and $\alpha_2 = \alpha_4 = \alpha_6 = \cdots$.*

Observe that the property $H(4)$ in Theorem 4.4 is sharp because of the following example.

Example 4.5 Consider $d\mu(t) = \chi_{[-1/3,2/3](t)dt}$. Then the associated weight sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$ can be represented by

$$\alpha_n = \frac{1}{\sqrt{3}}\sqrt{\frac{2^{n+2} + (-1)^{n+1}}{2^{n+1} + (-1)^n}}\sqrt{\frac{n+1}{n+2}}, \quad n \in \mathbb{Z}_+.$$

Define $\alpha(x,y) : \sqrt{u}, \sqrt{x}, \sqrt{y}, \sqrt{x}, \sqrt{y}, \sqrt{x}, \sqrt{y}, \alpha_n$ $(n \geq 7)$. Then the following assertions hold:

(i) $W_{\alpha(u,x,y)}$ has property $H(1) \Longleftrightarrow u \leq x,\ y \leq x,\ y \leq \frac{152}{255} = \alpha_7$,

(ii) $W_{\alpha(u,x,y)}$ has property $H(2) \Longleftrightarrow u \leq y \leq \frac{210902}{354135} \leq x \leq \frac{152}{255}$

(iii) $W_{\alpha(u,x,y)}$ has property $H(3) \Longleftrightarrow u < y < x \leq \delta$, where δ is the unique root of

$$g(y) = -210902 + 97279975y - 324622650y^2 + 271804500y^3$$

and, in fact, $\delta \approx 0.00218388...$

Corollary 4.6 *Suppose W_α has property $H(3)$. If $\alpha_{2n} = \alpha_{2n+2}$ and $\alpha_{2n+1} = \alpha_{2n+3}$ for some $n \in \mathbb{Z}_+$, then $\alpha_{2n} = \alpha_{2n+2} = \alpha_{2n+4}$ and $\alpha_{2n-1} = \alpha_{2n+1} = \alpha_{2n+3}$.*

We leave to the interested reader the formulation of the results analogous to Theorem 4.2 and Corollary 4.3 in their versions for the properties $\widetilde{H}(n)$. These follow easily upon noting that if W_α has some property $\widetilde{H}(n)$, then the restriction $W_\alpha|_{\vee\{e_i\}_{i=1}^\infty}$ has property $H(n)$. Observe also that the combination of properties $H(2)$ and $\widetilde{H}(2)$ is equivalent to 2-hyponormality, and thus we may recapture the result of Curto in [2] from the two limited left- and right-propagation results.

Theorem 4.7 *Let $\alpha(x)$ be a weight sequence given by*

$$\alpha(x) : \sqrt{x}, \sqrt{2/3}, \sqrt{2/3}, \sqrt{4/5}, \alpha_k = \sqrt{\frac{k+1}{k+2}} \quad (k \geq 4)$$

with x a positive real variable . Suppose $n \geq 3$. Then there exists $\delta_n \in (0, 2/3)$ with $\delta_3 \geq \delta_4 \geq \cdots$ such that the weighted shift $W_{\alpha(x)}$ has property $H(n)$ for any $x \in (0, \delta_n]$ but does not have property $H(n)$ for any $x \in (\delta_n, 2/3]$.

It is natural to ask what propagation results, if any, arise from the combination of some property $H(n)$ and $\alpha_{2k-1} = \alpha_{2k}$. Theorem 4.7 theorem shows that property $H(n)$ does not yield (further) flatness for any n. With this background the following extended problem is natural.

Problem A. Describe all weighted shifts W_α with property $H(n)$ such that $\alpha_1 = \alpha_2$.

We consider Problem A above with a weight sequence $\alpha(x)$ given by

$$\alpha(x) : \sqrt{x}, \sqrt{2/3}, \sqrt{2/3}, \sqrt{4/5}, \alpha_k = \sqrt{\frac{k+1}{k+2}}, \qquad k \geq 4, \tag{4.1}$$

where x is a positive real variable. For $n \in \mathbb{N}$, define

$$\delta_n := \sup\{x \in (0, 2/3] : W_{\alpha(x)} \text{ has property } H(n)\}. \tag{4.2}$$

It follows from Theorem 4.4 in [9] that there exists a family $\delta_n \in (0, 2/3)$ with $\delta_3 \geq \delta_4 \geq \cdots$ such that the weighted shift $W_{\alpha(x)}$ has property $H(n)$ for any $x \in (0, \delta_n]$ but does not have property $H(n)$ for any $x \in (\delta_n, 2/3]$. In fact, the exact value of δ_n remained open in [9]. But we obtain a formula to yield δ_n for any $n \in \mathbb{N}$ in the following theorem.

Theorem 4.8 *Let $\alpha(x)$ and δ_n be as in* (4.1) *and* (4.2)*. Then the following assertions hold.*

(i) $W_{\alpha(x,\frac{2}{3},\frac{2}{3})}$ *has property $H(1)$ if and only if $x \leq \frac{2}{3}$,*

(ii) $W_{\alpha(x,\frac{2}{3},\frac{2}{3})}$ *has property $H(2)$ if and only if $x \leq \frac{2}{3}$. (Observe that therefore $W_{\alpha(x,\frac{2}{3},\frac{2}{3})}$ has property $H(1)$ if and only if $W_{\alpha(x,\frac{2}{3},\frac{2}{3})}$ has property $H(2)$ if and only if $x \leq \frac{2}{3}$. Alternatively, $\delta_1 = \delta_2 = 2/3$.)*

(iii) *For* $n \geq 3$, $W_{\alpha(x,\frac{2}{3},\frac{2}{3})}$ *has property* $H(n)$ *if and only if* $x \in (0, \delta_n]$, *where*

$$\delta_n = \frac{933120(n+1)^2 g_1(n)}{n(n+2)\left(g_2(n) + 14695356n^4 - 9483192n^3 + g_3(n)\right)}, \quad n \geq 3,$$

where

$$\begin{aligned} g_1(n) &= n^6 + 6n^5 + 13n^4 + 12n^3 + 4n^2 + 288, \\ g_2(n) &= n^{16} + 16n^{15} + 109n^{14} + 406n^{13} + 862n^{12} + 880n^{11} + 10130n^{10} + 101828n^9 \\ &\quad + 336581n^8 + 220616n^7 + 833825n^6 + 8869142n^5, \\ g_3(n) &= 7762176n^2 + 63742464n + 517570560. \end{aligned}$$

Note that Theorem 4.8 is a solution of Problem A in the case of weight sequence with Bergman tail. The reader can solve Problem A in other operator models; for example, the weighted shifts with recursive weight whose construction will be discussed in Section 6.

5 A three weights completion problem

Let $\alpha_0, \alpha_1, \alpha_2$ be positive real numbers with $\alpha_0 < \alpha_1$. In this section we discuss a *Hamburger completion problem* with three weights $\alpha_0, \alpha_1, \alpha_2$ as the initial data: the goal is to find a weight sequence $\widehat{\alpha}$ extending $\alpha_0, \alpha_1, \alpha_2$ such that the associated weighted shift $W_{\widehat{\alpha}}$ is Hamburger-type. (Note that the restriction $\alpha_0 < \alpha_1$ is harmless, as $\alpha_0 \leq \alpha_1$ is forced by property $H(1)$, and if $\alpha_0 = \alpha_1$ then the flatness result in Theorem 4.2 forces all weights equal.) For this purpose, we consider two possibilities:

1° *the initial data give rise to a completion moment sequence which is Hamburger,*

2° *the initial data give rise to a completion moment sequence which is Hamburger with all positive moments.*

In the presence of 2° we may define a Hamburger-type weighted shift in the usual way (abiding by our assumption that weights are positive), but 1° is not enough for this. There are two approaches to trying to find some completion at least satisfying 1°, and we turn to the first, leaving the second for remarks at the end of the section.

We may imitate the Curto-Fialkow construction ([3, p.231]). This is most easily described in terms of weights, so assume $\alpha_0 < \alpha_1$ as above. Note also $\alpha_2 \geq \alpha_1$ is the Stampfli case where a subnormal completion is possible ([14]). Set

$$s_0 = \frac{\psi_1 - \sqrt{\psi_1^2 + 4\psi_0}}{2}, \qquad s_1 = \frac{\psi_1 + \sqrt{\psi_1^2 + 4\psi_0}}{2}, \qquad \rho = \frac{s_1 - \alpha_0^2}{s_1 - s_0}, \tag{5.1}$$

where

$$\psi_0 = -\frac{\alpha_0^2 \alpha_1^2 (\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}, \qquad \psi_1 = \frac{\alpha_1^2 (\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}, \tag{5.2}$$

and define $\mu = \rho\delta_{s_0} + (1-\rho)\delta_{s_1}$. We will claim that μ yields the correct moments (hence at least initial weights) to match the initial data, and so it induces a Hamburger completion at least in the sense of 1°.

There are things to check (since we don't assume $\alpha_1 < \alpha_2$). First, $s_0, s_1 \in \mathbb{R}$, because the expression inside the defining square root is a quadratic in α_2^2 which, after the substitution $\alpha_1 = \alpha_0 + \epsilon$ with $\epsilon > 0$, has no real zeros. Since we know s_0 is real, and hence this quantity is positive for α_2 large enough (the Stampfli case), it must always be positive. So $s_0, s_1 \in \mathbb{R}$. (Note that we expect that for the Hamburger-type case s_0 may be negative, but this is in fact for us the case of interest.)

We also need ρ (real and) satisfying $0 \le \rho \le 1$. That ρ is real is easy.

For $\rho \ge 0$, one checks easily that $\rho(k\alpha_0, k\alpha_1, k\alpha_2) = k\rho(\alpha_0, \alpha_1, \alpha_2)$, where $\rho := \rho(\alpha_0, \alpha_1, \alpha_2)$ is as in (5.1). So it suffices to consider the case in which $\alpha_0 = 1$. Then after substituting $\alpha_1^2 = 1 + \epsilon$ (with $\epsilon > 0$) we must check

$$-1 + \alpha_2^2(1+\epsilon) - 3\epsilon + \sqrt{(1+\epsilon)(1 + 5\epsilon + 4\epsilon^2 + \alpha_2^4(1+\epsilon) - 2\alpha_2^2(1+3\epsilon))} > 0,$$

and via the usual technique of moving $-1 + \alpha_2^2(1+\epsilon) - 3\epsilon$ to the other side, squaring both sides, and simplifying, this turns out to be correct. For $\rho \le 1$, we have $\rho = \frac{s_1 - \alpha_0^2}{s_1 - s_0}$ and we obviously need $\alpha_0^2 \ge s_0$. If $s_0 > 0$ we are in the Stampfli subnormal case and know $\rho \le 1$; if $s_0 < 0$, obviously $\alpha_0^2 > s_0$. Therefore we may define the measure $\mu = \rho\delta_{s_0} + (1-\rho)\delta_{s_1}$.

By a direct computation, we see easily that the pair of conditions $p \le q$ and $a < \frac{q}{p+q}$ is equivalent to $\alpha_0 \le \alpha_2 < \alpha_1$. If $\alpha_2 < \alpha_0 < \alpha_1$, then γ_n can be negative for some $n \in \mathbb{N}$; cf., see "Moreover" part of Proposition 5.1.

Proposition 5.1 *Let $\alpha : \alpha_0, \alpha_1, \alpha_2$ be positive real numbers with $\alpha_0 < \alpha_1$. Then there exists a (2 atomic) measure $\mu = \rho\delta_{s_0} + (1-\rho)\delta_{s_1}$ with $0 \le \rho \le 1$, where ρ, s_0, s_1 are as in* (5.1) *and* (5.2), *and a sequence $\{\widehat{\gamma}_n\}_{n=0}^{\infty} \subset \mathbb{R}$ with $\widehat{\gamma}_j = \gamma_j$ $(j = 0, 1, 2)$ such that*

$$\widehat{\gamma}_n = \int_{\mathbb{R}} t^n \, d\mu, \qquad n \in \mathbb{Z}_+.$$

Moreover, if $\alpha_0 \le \alpha_2 < \alpha_1$, we can take a sequence $\widehat{\alpha} = \{\widehat{\alpha}_n\}_{n=0}^{\infty} \subset \mathbb{R}$ with $\widehat{\alpha}_j = \alpha_j$ $(j = 0, 1, 2)$ such that $\widehat{\gamma}_n = \widehat{\alpha}_0^2 \cdots \widehat{\alpha}_{n-1}^2$ for $n \in \mathbb{Z}_+$.

We remark that, in the case of "Moreover" part of Proposition 5.1, it is easy to verify that the weights satisfy the recursion

$$\widehat{\alpha}_n^2 = \psi_1 + \frac{\psi_0}{\widehat{\alpha}_{n-1}^2} \qquad n \ge 1. \tag{5.3}$$

Definition 5.2 Given initial positive weights $\alpha : \alpha_0 \le \alpha_2 < \alpha_1$, we will denote the Hamburger completion sequence of weights arising via the construction captured in Proposition 5.1 by $(\alpha_0, \alpha_1, \alpha_2)^H$.

Note that in this case we do not allow $\alpha_0 < \alpha_1 < \alpha_2$ for which there is a (Hausdorff) Stampfli completion, nor do we allow $\alpha_0 < \alpha_1 = \alpha_2$ for which there is a (flat) Hausdorff completion, and recall that $\alpha_0 \leq \alpha_1$ is required by property $H(1)$.

We may then obtain the following; with a slight abuse of previous language, we will say that a moment sequence has some property $H(n)$ with the obvious meaning.

Theorem 5.3 *Let $\alpha_0, \alpha_1, \alpha_2$ be positive real numbers. Then the condition $\alpha_0 \leq \alpha_2 < \alpha_1$ is equivalent to the assertion that the real numbers $\alpha_0, \alpha_1, \alpha_2$ produce a Hamburger completion $(\alpha_0, \alpha_1, \alpha_2)^H$ with strictly positive weights but whose associated weighted shift $W_{(\alpha_0,\alpha_1,\alpha_2)^H}$ is not subnormal.*

Corollary 5.4 *Suppose $1 \leq y < x$. Then $(1, \sqrt{x}, \sqrt{y})^H$ has a backward 2-step Hamburger-type extension if and only if $y < \frac{2x-1}{x}$.*

6 Backward extensions and perturbations

Suppose W_α is a Hamburger-type weighted shift. Let $\alpha(x) : x, \alpha_0, \alpha_1, \cdots$ be a backward 1-step extension of the weight sequence α. It turns out as we see next that such a "backward 1-step" extension is not, perhaps, the natural thing to study.

Proposition 6.1 *Suppose W_α is a Hamburger-type weighted shift such that for some $x > 0$, $W_{\alpha(x)}$ is a Hamburger-type weighted shift. Then W_α is subnormal. In this case, any backward 1-step extension $W_{\alpha(x)}$ of W_α is a Hamburger-type shift if and only if $W_{\alpha(x)}$ is subnormal.*

Note that the proof of Proposition 6.1 shows that if W_α has property $H(n)$ (respectively, is Hamburger-type), then any backward 1-step extension has property $\widetilde{H}(n)$ (respectively, has property $\widetilde{H}(\infty)$). The converse is equally easy.

The same sort of approach yields the following proposition.

Proposition 6.2 *Let W_α be a Hamburger-type weighted shift with property $\widetilde{H}(n)$ for some $n \in \mathbb{N}$. Suppose $M_n(1)$ is strictly positive. Then there exists $x \in \mathbb{R}_+^0$ such that $W_{\alpha(x)}$ has property $\widetilde{H}(\infty)$ and $H(n)$.*

The results above impel us to study backward extensions of "even" length. Consider now a backward extension of length two: W_α with $\alpha :\ \alpha_0, \alpha_1, ...$ and $\alpha(x, y) : x, y, \alpha_0, \alpha_1, ...$ yielding the corresponding weighted shift $W_{\alpha(x,y)}$. The results here then work in any determinate case (for an unbounded densely defined shift) but it may be interesting to consider extensions in the indeterminate case. The following is completely parallel to portions of [6, Lemma 2.1], and see also [2, Prop. 8].

Theorem 6.3 *Suppose W_α is a Hamburger-type weighted shift with $\alpha = \{\alpha_i\}_{i=0}^\infty$. Let $\alpha(x,y):\ x, y, \alpha_0, \alpha_1, \cdots$ be a backward 2-step extension of α, where $x, y \in \mathbb{R}_+^0$. Then $W_{\alpha(x,y)}$ is a Hamburger-type weighted shift if and only if the following four conditions hold:*

$$\frac{1}{t^2} \in L^1(\mu),\ \int_{\mathbb{R}} \frac{1}{t} d\mu(t) > 0,\ \ y = \frac{1}{\left(\int_{\mathbb{R}} \frac{1}{t} d\mu(t)\right)^{\frac{1}{2}}},\ \ 0 < x \le \left(\frac{\int_{\mathbb{R}} \frac{1}{t} d\mu(t)}{\int_{\mathbb{R}} \frac{1}{t^2} d\mu(t)}\right)^{\frac{1}{2}}, \tag{6.1}$$

where μ is the Hamburger measure associated with W_α. In this case, $W_{\alpha(x,y)}$ has Hamburger measure ν defined by

$$d\nu = \lambda\delta_0 + x^2y^2 \cdot \frac{1}{t^2} d\mu, \qquad \lambda = 1 - x^2y^2 \int_{\mathbb{R}} \frac{1}{t^2} d\mu.$$

Remark 6.4 The weight y in Theorem 6.3 is completely and uniquely determined by α (or equivalently μ). Further, if $\lambda = 0$ (equivalently, $\hat{\nu}(\{0\}) = 0$) x is also uniquely determined at its maximum possible value.

We may generalize to longer backward extensions in a familiar way (cf., for example, Theorem 3.5 of [6]).

Theorem 6.5 *Let W_α be a Hamburger-type weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^\infty$ and let μ be the corresponding Hamburger measure. Suppose $x_1, x_2, \ldots, x_{2n-1}$, and x_{2n} are positive. Then $W_{\alpha(x_{2n}, x_{2n-1}, \cdots, x_2, x_1)}$ is a Hamburger-type weighted shift if and only if the following conditions hold*

(i) $\frac{1}{t^{2n}} \in L^1(\mu),\quad \int_{\mathbb{R}} \frac{1}{t^{2j-1}} d\mu > 0,\quad 1 \le j \le n,$

(ii) $x_j = \left(\frac{\|\frac{1}{t^{j-1}}\|_{L^1(\mu)}}{\|\frac{1}{t^j}\|_{L^1(\mu)}}\right)^{\frac{1}{2}}$ *for* $1 \le j \le 2n-1$, *and*

(iii) $0 < x_{2n} \le \left(\frac{\|\frac{1}{t^{2n-1}}\|_{L^1(\mu)}}{\|\frac{1}{t^{2n}}\|_{L^1(\mu)}}\right)^{\frac{1}{2}}.$

Further, in this case the Hamburger measure ν_{-2n} for $W_{\alpha(x_{2n}, x_{2n-1}, \cdots, x_2, x_1)}$ is

$$d\nu_{-2n} = \lambda_{-2n}\delta_0 + x_1^2 \cdots x_{2n}^2 \cdot \frac{1}{t^{2n}} d\mu, \qquad \lambda_{-2n} = 1 - x_1^2 \cdots x_{2n}^2 \int_{\mathbb{R}} \frac{1}{t^{2n}} d\mu.$$

We next give an example of a Hamburger-type weighted shift which is not Hausdorff-type but which allows Hamburger-type backward extensions.

Example 6.6 Let us consider a measure of the form

$$d\mu_\epsilon = \epsilon\delta_{-\epsilon} + \chi_{[\epsilon,1]}(t)dt, \quad 0 < \epsilon < 1.$$

Since

$$\gamma_n = \int_{\mathbb{R}} t^n \, d\mu = (-1)^n \epsilon^{n+1} + \frac{1}{n+1}\left(1 - \epsilon^{n+1}\right), \qquad n \in \mathbb{Z}_+, \tag{6.2}$$

it is obvious that $\gamma_{2k} > 0$ for all $k \in \mathbb{Z}_+$. By (6.2),

$$\gamma_{2k+1} = \frac{2k+3}{2k+2}\left(\frac{1}{2k+3} - \epsilon^{2k+2}\right) > 0$$

for any ϵ such that $0 < \epsilon < e^{-1}$. Let μ_ϵ be a moment measure with $0 < \epsilon < e^{-1}$ and let W_α be the associated weighted shift. Then W_α satisfies the four conditions of (6.1) in Theorem 6.3. But it does not satisfy Theorem 6.5(i); indeed,

$$\int t^{-3} \, d\mu_\epsilon = -\frac{1}{2}\left(\epsilon^{-2} + 1\right) < 0.$$

Hence W_α is Hamburger-type backward 2-step extendable but not a Hamburger-type backward 4-step extendable weighted shift.

We now consider the backward extension of a recursively generated Hamburger-type weighted shift.

Theorem 6.7 *Let a, b, and c with $0 < a \le c < b$ be given. Suppose $W_{(\sqrt{a},\sqrt{b},\sqrt{c})^H}$ is a recursively generated Hamburger-type weighted shift. Let*

$$\alpha : \sqrt{x_{2n}}, \sqrt{x_{2n-1}}, ..., \sqrt{x_1}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^H$$

be a backward $2n$-step extension of $(\sqrt{a}, \sqrt{b}, \sqrt{c})^H$, where $x_j \in \mathbb{R}_+^0$, $1 \le j \le 2n$. Then

$$W_\alpha \textit{ is a Hamburger-type weighted shift} \Longleftrightarrow \begin{cases} W_\alpha \textit{ has property } H(3), & n = 1, \\ W_\alpha \textit{ has property } H(4), & n \ge 2. \end{cases}$$

We may now turn to perturbations. Theorem 6.3 shows that a non-zero perturbation in the weights α_0 and α_1 which yields a shift with property $H(\infty)$ must be, in fact, one fixing α_1, and decreasing α_0 (view W'_α as a backward 2-step extension of $W_\alpha|_{\vee\{e_i\}_{i=2}^\infty}$). What follows is the analogue of the rest of Theorem 2.1 of [5].

Theorem 6.8 *No finite perturbation of the weights of some Hamburger-type weighted shift W_α that actually changes some α_j, $j \ge 1$, can yield a Hamburger-type weighted shift $W_{\alpha'}$.*

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